Lecture 1: Galois Theory

- Fundamental Theorem
- e^x, $\log x$ not algebraic

  Same tools show $\gamma(t,x) = \int_0^x s^{t-1} e^{-s} ds$

  Although γ satisfies $\frac{\partial^2 \gamma}{\partial x^2} - \frac{t-1-x}{x} \frac{\partial \gamma}{\partial x} = 0$

  γ satisfies no $p(t, x, \gamma, \frac{\partial \gamma}{\partial t}, \ldots, \frac{\partial^n \gamma}{\partial t^n}) = 0$
- $y(x) = \cos x$ satisfies $y'' + y = 0$

  but $\sec x = \frac{1}{\cos x}$ satisfies no LDE

  Same tools $\Rightarrow$ $\frac{1}{\text{Bessel func}}$ satisfies no LDE.

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k - field of characteristic 0

$' : k \to k$ $\quad (x+y)' = x' + y'$ $\quad (xy)' = x'y + xy'$

$(k, ')$ = <u>differential field</u>

<u>Ex:</u> k = meromorphic funcs on domain $G \subset \mathbb{C}$

$' = \frac{d}{dx}$

<u>Linear Differential Equation</u> (LDE)

$$Y' = AY \qquad A \in gl_n(k)$$

$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_0 y = 0 \Leftrightarrow Y' = AY$

$$\begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}' = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & \cdots & & -a_{n-1} \end{pmatrix} \begin{pmatrix} y \\ y' \\ \vdots \\ y^{(n-1)} \end{pmatrix}$$

Galois Theory $<$ Splitting Field / Automorphism Group

<u>Picard - Vessiot Extension</u> K of k for $Y' = AY$

- $K = k(y_{11}, \dots, y_{nn})$ $\quad Y = (y_{ij}) \in GL_n(K)$, $\; Y' = AY$
- $\mathcal{C}_K = \mathcal{C}_k$ $\qquad \mathcal{C}_F = \{ c \in F \mid c' = 0 \}$

<u>FACT</u>: $\mathcal{C}_k$ algebraically closed $\Rightarrow \exists!$ PV extension

<u>Ex</u>: $\mathbb{C}(x)$ $\; x' = 1 \quad A \in GL_n(\mathbb{C}(x))$

At non-singular point $\exists$ local analytic solns.

<u>THM</u>: $Y' = AY \quad A \in \mathfrak{gl}_n(k) \quad K = k(\{y_{ij}\})$ PV extension. $\mathcal{C} = \mathcal{C}_k$

- $DGal(K/k) = \{ \sigma : K \to K \mid \sigma|_k = id,\ \sigma = \text{AUTOMORPHISM},\ \sigma(z') = (\sigma(z))' \}$

  $\sigma \in DGal \Rightarrow \sigma(Y) = Y C_\sigma \quad C_\sigma \in GL_n(\mathcal{C})$

  $DGal \hookrightarrow GL_n(\mathcal{C})$ IMAGE IS <u>ZARISKI CLOSED</u>

- $H^{Z\text{-closed}} \subset DGal \leftrightarrow k \subset F^{\text{DIFF. FIELD}} \subset K$

  $H \mapsto K^H = \{ z \in K \mid \sigma(z) = z \ \forall \sigma \in DGal \}$

  $F \mapsto \{ \sigma \in DGal \mid \sigma(z) = z \ \forall z \in F \}$

- $H^{Z\text{-closed}} \triangleleft G \Leftrightarrow K^H$ is a PV-ext of k
- $\text{tr. deg.}(K/k) = \dim_{\mathcal{C}} DGal(K/k)$

Ex: $n=1$ $y' = ay$ $a \in k^*$

$$\mathrm{DGal} \subset GL_1(\mathbb{C}) = \mathbb{C}^*$$

Algebraic subgps $= \begin{cases} GL_1(\mathbb{C}) \\ \mathbb{Z}/m\mathbb{Z} = \{ c \in \mathbb{C} \mid c^m = 1 \} \end{cases}$

Prop $y' = ay$ y algebraic over $k \Leftrightarrow \exists m \in \mathbb{N} \setminus \{0\}$ $y^m \in k$ $\Leftrightarrow \exists f \in k$ s.t. $ma = f'/f$

Pf: y alg $/k \Rightarrow \mathrm{Dgal} = \mathbb{Z}/m\mathbb{Z}$

$\sigma \in \mathrm{Dgal}$, $\sigma(z^m) = c^m z^m = z^m$

$\Rightarrow z^m \in k \Rightarrow \frac{(z^m)'}{z^m} = ma \in k$

Prop $y' = a \in k$, y algebraic$/k \Leftrightarrow \exists z \in k$, $z' = a$

Pf: $\begin{pmatrix} y \\ 1 \end{pmatrix}' = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \begin{pmatrix} y \\ 1 \end{pmatrix}$ $K = k(y)$

$\sigma \in \mathrm{Dgal}$ $\sigma(y) = y + c_\sigma$

$\mathrm{Dgal} \subset (\mathbb{C}, +) = \left\{ \begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix} \middle| c \in \mathbb{C} \right\}$

Alg. subgps : $\begin{cases} (\mathbb{C}, +) \\ (0) \end{cases}$

$\therefore$ $y' = ay$ y alg $\Rightarrow \mathrm{Dgal} = (0) \Rightarrow k(y) = k \Rightarrow y \in k$.

Ex: ① e^x alg $/\mathbb{C}(x) \Rightarrow m = \frac{f'}{f} \Rightarrow m = \sum \frac{c_i}{(x - \alpha_i)}$ $c_i, \alpha_i \in \mathbb{C}$ #

② $\log x$ alg$/\mathbb{C}(x) \Rightarrow \exists f \in \mathbb{C}(x)$ $f' = \frac{1}{x}$ #

Prop: $y_0' = a_0 \; \cdots \; y_{n-1}' = a_{n-1}$

$y_0, \ldots, y_{n-1}$ alg. dependent over k $\Leftrightarrow$ $\exists c_0, \ldots, c_{n-1}$ NAZ $\sum c_i y_i \in k$

$\Leftrightarrow \sum c_i a_i = f', \; f' \in k$

Proof: $\Rightarrow$ $D\,\mathrm{Gal} \subsetneq (\mathcal{C}, +)^n$
$\Rightarrow \exists d_0, \ldots, d_{n-1}$ s.t. $\sum c_i X_i$ vanishes on $D\,\mathrm{Gal}$
$\Rightarrow \sigma(\sum d_i y_i) = \sum d_i (y_i + c_i) = \sum d_i y_i$
$\Rightarrow \sum d_i y_i \in k.$

INCOMPLETE γ - FUNCTION

$\gamma_0 = \gamma(x, t) = \int_0^x s^{t-1} e^{-s} ds, \quad \theta = s^{t-1} e^{-s} = e^{-s + (t-1)\log s}$
$= \int_0^x \theta \, ds$

$\gamma_1 = \frac{\partial \gamma}{\partial t} = \int_0^x \log s \, \theta \, ds, \ldots$

$\gamma_i = \frac{\partial^i \gamma}{\partial t^i} = \int_0^x (\log s)^i \theta \, ds$

Let $\mathcal{C} = \overline{\mathbb{C}(t)}$, $k = \mathcal{C}(x, \log x, e^{-x + (t-1)\log x} = \theta)$
$k = \frac{\partial}{\partial x}$ field

$\frac{\partial \gamma_0}{\partial x} = \theta, \quad \frac{\partial \gamma_1}{\partial x} = (\log x)\theta, \; \cdots \; \frac{\partial \gamma_n}{\partial x} = (\log x)^n \theta$

γ - satisfies $p(x, t, \gamma_0, \ldots, \gamma_n) = 0$
$\frac{\partial \gamma_i}{\partial x} \in k$

$\Leftrightarrow \exists c_0, \ldots, c_n \in \mathcal{C} \quad \sum c_i (\log x)^i \theta = f'$
$f \in k.$

KOLCHIN-OSTROWSKI THEOREM

$a_1, \dots, a_n, b_1, \dots, b_m \in k$

$y_1 \dots y_n, z_1 \dots z_m \in K$ s.t. $y_i' = a_i \quad z_j' = b_j \cdot z_j$

$K = k(y_1, \dots, y_m, z_1 \dots z_m)$

If $\operatorname{trdeg}(K/k) < n+m \Rightarrow$

$$\text{Either} \begin{cases} \exists c_1 \dots c_m \in \mathcal{C} \text{ (NAZ)}, \sum c_i y_i \in k \\ \quad \text{or} \\ \exists n_1 \dots n_m \in \mathbb{Z} \text{ (NAZ)}, \prod z_i^{n_i} \in k \end{cases}$$

Pf: Assume such c_i, n_j do not exist

$G_1 = \operatorname{DGal}(k(y_1 \dots y_n)/k) = (\mathcal{C}, +)^n$

$G_2 = \operatorname{DGal}(k(z_1 \dots z_m)/k) = (\mathcal{C}^*, \cdot)^m$

$H = \operatorname{DGal}(K/k) \subsetneq (\mathcal{C}, +)^n \times (\mathcal{C}, \cdot)^m$ (since $\dim_{\mathcal{C}}(H) < n+m$)

$\pi_1 : G \to (\mathcal{C}, +)^n = \operatorname{DGal}(k(y_1 \dots y_n)/k)$ is surjective

$\pi_2 : G \to (\mathcal{C}^*, \cdot)^m = \operatorname{DGal}(k(z_1 \dots z_m)/k)$ is surjective

(Weak) Goursat Lemma: $H \subset G = G_1 \times G_2 \quad \pi_1 : H \to G_1 \quad \pi_2 : H \to G_2$

Let $\{0\} \times N_2 = \ker \pi_1$, $N_2 \triangleleft G_2$, $N_1 \times \{1\} = \ker \pi_2 \quad N_1 \triangleleft G_1$

Then $G_1/N_1 \simeq G_2/N_2$

Apply to G, G_1, G_2 above $\Rightarrow (\mathcal{C}, +)^n/N_1 \simeq (\mathcal{C}^*, \cdot)^m/N_2$

$$\left.\begin{array}{l} \text{Elements of } (\mathcal{C}, +)^n/N_1 \text{ are unipotent} \\ \quad " \quad " \quad (\mathcal{C}, \cdot)^m/N_2 \text{ are semisimple} \end{array}\right\} \Rightarrow \begin{array}{l} N_1 = G_1 \\ N_2 = G_2 \end{array}$$

So $\{0\} \times G_2, G_1 \times \{1\} \subset H \Rightarrow H = G_1 \times G_2$ #

Galois Theory of Polynomial Eqns:

$k = \mathbb{Q}$

$X^2+1 \Rightarrow V = \{c \mid c^2+1=0\}$ defined$/\mathbb{Q}$ $\quad V(\mathbb{Q}) = \emptyset$

$X^2-1 \Rightarrow G = \{c \mid c^2-1=0\}$ " " $\quad G(\mathbb{Q}) = \{\pm 1\}$
Group.

$G \times V \to V \quad (g,v) \mapsto vg$ defined over $\mathbb{Q}$

$G(\overline{\mathbb{Q}}) \times V(\overline{\mathbb{Q}}) \to V(\overline{\mathbb{Q}})$

$\forall\, v, w \in V(\overline{\mathbb{Q}})\ \exists!\, g \in G \quad vg = w$

V is a Principal Homogenous Space for G

__Note:__ $V(\overline{\mathbb{Q}}) \simeq G(\overline{\mathbb{Q}})$

$$\overline{\mathbb{Q}}[X]/(X^2+1) \simeq \overline{\mathbb{Q}}[X]/(X^2-1)$$

__FACT__ $Y' = AY \quad A \in gl_n(k) \qquad K = k(\{y_{ij}\})$ PV extension
$G = \mathrm{DGal}(K/k)$

$$R = k[\{y_{ij}\}, \tfrac{1}{\det(y_{ij})}]$$

- R is the coordinate ring (over k) of a PHS for G
- $R \otimes \overline{k} = \overline{k}[G]$

The Galois action of $g \in G$ on R

$\Updownarrow$

$$\rho_g^*: \overline{k}[G] \to \overline{k}[G] \quad \text{where } \rho(g)(h) = h \cdot g$$

__NOTE:__ $R = \{y \in K \mid y \text{ satisfies a LDE}/k\}$

<u>Thm</u> (Harris–Sibuya)

If $y \neq 0$ and $1/y$ satisfy an LDE/k

$\Rightarrow y'/y$ alg./k

<u>Ex:</u> $\cos x$ satisfies $y''+y=0$

If $\sec x = \frac{1}{\cos x}$ satisfies LDE/$\mathbb{C}(x)$

$\Rightarrow -\tan x = \frac{\cos x}{\sin x}$ algebraic over $\mathbb{C}(x)$

$\Rightarrow \arctan x$ algebraic over $\mathbb{C}(x)$

$(\arctan x)' = \frac{1}{1+x^2} \in \mathbb{C}(x)$ but $\nexists f \in \mathbb{C}(x)$ s.t. $f' = \frac{1}{1+x^2}$

<u>Ex:</u> Bessel $y(x)$ satisfies $y'' + \frac{1}{x}y' + (1 - \frac{\nu^2}{x^2})y = 0$

if $\nu - \frac{1}{2} \notin \mathbb{Z} \Rightarrow y'/y$ not algebraic.

(see Exercises)

<u>Proof of Thm.</u>

Can assume k algebraically closed.

K = PV extension containing $y, \frac{1}{y}$

$G = \mathrm{DGal}(K/k)$

$y, \frac{1}{y} \in k[G] = k[g, \frac{1}{\det(g)}]$

Rosenlicht $\Rightarrow y = af$ $a \in k$ f = character of G

$f(g \cdot h) = f(g) \cdot f(h)$

$\sigma \in \mathrm{DGal} \Rightarrow \sigma(y) = a \cdot f(g \cdot \sigma) = af(g) \cdot f(\sigma)$

Can show $f(\sigma) \in \mathcal{C} \Rightarrow \sigma(y) = c_\sigma \cdot y \Rightarrow y'/y \in k.$